

LYAPUNOV ‘NON-TYPICAL’ POINTS OF MATRIX COCYCLES AND TOPOLOGICAL ENTROPY

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Abstract. It follows from Oseledec Multiplicative Ergodic Theorem (or Kingman’s Sub-additional Ergodic Theorem) that the set of ‘non-typical’ points for which the Oseledec averages of a given continuous cocycle diverge has zero measure with respect to any invariant probability measure. In strong contrast, for any Höder continuous cocycles over hyperbolic systems, in this article we show that either all ergodic measures have same Maximal Lyapunov exponents or the set of Lyapunov ‘non-typical’ points have full topological entropy and packing topological entropy. Moreover, we give an estimate of Bowen Hausdorff entropy from below.

1. INTRODUCTION

1.1. Lyapunov Exponents. Let f be an invertible map of a compact metric space X and let $A : X \rightarrow GL(m, \mathbb{R})$ be a continuous matrix function. One main object of interest is the asymptotic behavior of the products of A along the orbits of the transformation f , called cocycle induced from A : for $n > 0$

$$A(x, n) := A(f^{n-1}(x)) \cdots A(f(x))A(x),$$

and

$$A(x, -n) := A(f^{-n}(x))^{-1} \cdots A(f^{-2}(x))^{-1}A(f^{-1}(x))^{-1} = A(f^{-n}x, n)^{-1}.$$

The Maximal Lyapunov exponent (or simply, MLE) of $A : X \rightarrow GL(m, \mathbb{R})$ at one point $x \in X$ is defined as the limit

$$\chi_{max}(A, x) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \|A(x, n)\|,$$

if it exists. In this case x is called to be (forward) *Max-Lyapunov-regular* (simply, ML-regular). Otherwise, x is *Max-Lyapunov-irregular* (simply, ML-irregular, or called Lyapunov ‘nontypical’ point). By Kingman’s Sub-additional Ergodic Theorem (or Oseledec’s Multiplicative Ergodic Theorem), for any invariant measure μ and μ a.e. point x , MLE always exists and the function $\chi_{max}(A, x)$ is f -invariant. Define $\chi_{max}(A, \mu) = \int \chi_{max}(A, x) d\mu$. In particular, for any ergodic measure μ and μ a.e. point x , MLE always exists and is constant, equal to $\chi_{max}(A, \mu)$. Let $MLI(A, f)$ denote the set of all ML-irregular points, called ML-irregular set. Then it is of zero measure for any invariant measures.

In this paper we will mainly pay attention to the dynamical complexity of ML-irregular points, in the sense of entropy.

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1.2. Results. The set of all invariant measures and the set of all ergodic invariant measures are denoted by $\mathcal{M}_f(X)$ and $\mathcal{M}_f^e(X)$, respectively. An ergodic measure μ is called Lyapunov-minimizing for cocycle A , if

$$\chi_{\max}(\mu, A) = \inf_{\nu \in \mathcal{M}_f(X)} \chi_{\max}(\nu, A).$$

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Theorem 1.1. *Let f be a C^1 diffeomorphism of a compact Riemannian manifold M and $X \subseteq M$ be a topologically mixing locally maximal hyperbolic invariant subset. Let $A : X \rightarrow GL(m, \mathbb{R})$ be a Höder continuous matrix function. Then either*

- (1) all ergodic measures supported on X have same MLE w.r.t. A ; or*
- (2) $MLI(A, f) \cap X$ has full topological entropy of X and has full packing topological entropy of X and moreover, its Bowen Hausdorff entropy can be estimated from below by*

$$\sup_{\mu \in \mathcal{M}_f^e(X)} \{h_\mu(f) \mid \mu \text{ is not Lyapunov minimizing for } A\},$$

where $h_\mu(f)$ denotes the metric entropy of μ . In particular, if the (unique) maximal entropy measure of $f|_X$ is not Lyapunov minimizing, then $MLI(f) \cap X$ has full Bowen Hausdorff entropy of X .

It is well-known that the maximal entropy measure of system restricted on any topologically mixing locally maximal hyperbolic set exists and unique, see [8] (or [17]).

Remark 1.2. From [20, Theorem 4] we know for any uniquely ergodic system, there exists some continuous matrix cocycle whose Lyapunov-irregular set is a dense set of second Baire category. So, it is admissible to satisfy that all ergodic measures have same Lyapunov spectrum and simultaneously Lyapunov-irregular points form a set of second Baire category. On the other hand, if $m = 1$, the above phenomena of [20] naturally does not happen for any dynamical system. Let us explain more precisely. If $m = 1$, the Lyapunov exponent can be written as Birkhoff ergodic average

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x))$$

where $\phi(x) = \log \|A(x)\|$ is a continuous function. If all ergodic measures have same Lyapunov spectrum, then by Ergodic Decomposition theorem so do all invariant measures and thus by weak* topology, the limit $\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x))$ should exist at every point $x \in X$ and equal to the given spectrum. Moreover, the case of $m = 1$ is in fact to study Birkhoff ergodic average and it has been studied for systems with specification or its variants by many authors, see [7, 3, 40, 41] and reference therein. As said in [4] that the study of Lyapunov exponents lacks today a satisfactory general approach for non-conformal case, since a complete understanding is just known for some cases such as requiring a clear separation of Lyapunov directions or some number-theoretical properties etc, see [4] and its references therein. Our result gives one such characterization for non-conformal maps on ‘non-typical’ points of Lyapunov exponents.

Remark 1.3. From [29, 13] and its references therein, we know for generic continuous function, there is a unique maximizing (resp., minimizing, sometimes called Ergodic Optimization) measure with zero entropy. Moreover, many people aim to show that ‘most’ functions are optimized by measures supported on a periodic orbit.

For example, from [45] we know for hyperbolic systems (Axiom A), any Lipschitz function optimized by a periodic orbit measure can be perturbed to be stably optimized by this periodic orbit measure; and from [15] for expanding systems, the subset of functions uniquely optimized by measures supported on a periodic point is open and dense in the space of Höder continuous functions (in topology of Höder norm). These discussion are for the particular case of matrix functions with $m = 1$, and it is still an open question for the characterization of Lyapunov maximizing (resp., minimizing) measure for matrix functions with $m \geq 2$. However, above analysis gives us some positive direction to conjecture or believe that for ‘most’ cocycles of $m \geq 2$, the minimizing measure should have metric entropy less than full entropy. If this is true, then for ‘most’ cocycles the Bowen Hausdorff entropy of Theorem 1.1 is also full and same as another two entropy. Moreover, on one hand, it is unknown whether minimizing measure must exist naturally. If not, the Bowen Hausdorff entropy of Theorem 1.1 is also full. On the other hand, we know that for hyperbolic system, there are infinite ergodic measure arbitrarily close to the full entropy (see entropy-dense property below), thus for a matrix function with $m \geq 2$, if its minimizing measure exists but just forms a finite set, then the Bowen Hausdorff entropy of Theorem 1.1 is full. All in all, the better estimate of the Bowen Hausdorff entropy for Lyapunov ‘non-typical’ points are based on the advance of Lyapunov Ergodic Optimization (there are few such results, just recently some people started to study, for example, see [6]).

As a particular case of Theorem 1.1 we have a consequence on topological entropy for the derivative cocycle of all hyperbolic systems. Let $MLI(f) := MLI(Df, f)$. It is called ML-irregular set of system f .

Theorem 1.4. *Let f be a $C^{1+\alpha}$ diffeomorphism of a compact Riemannian manifold M and $X \subseteq M$ be a topologically mixing locally maximal invariant subset. Then either*

- (1) *all ergodic measures supported on X have same MLE; or*
- (2) *ML-irregular set $MLI(f) \cap X$ has full topological entropy of X and has full packing topological entropy of X and moreover, its Bowen Hausdorff entropy can be estimated from below by*

$$\sup_{\mu \in \mathcal{M}_f^e(X)} \{h_\mu(f) \mid \mu \text{ is not Lyapunov minimizing for } Df\}.$$

The motivations to obtain our main theorems are from some ideas of recent works of [19, 25, 41, 16]. In section 2 we first give some preparation and then in section 3 we give the proof.

2. PRELIMINARIES

2.1. Entropy. Let (X, d) be a compact metric space with Borel σ -algebra $\mathfrak{B}(X)$ and let $f : X \rightarrow X$ be a continuous map. Let $\mathcal{M}(X)$ denote the space of all probability measures supported on X . The set of all invariant measures and the set of all ergodic invariant measures are denoted by $\mathcal{M}_f(X)$ and $\mathcal{M}_f^e(X)$, respectively. For $x, y \in X$ and $n \in \mathbb{N}$, let

$$d_n(x, y) = \max_{0 \leq i \leq n-1} d(f^i(x), f^i(y)).$$

Let $x \in X$. The dynamical open ball $B_n(x, \varepsilon)$ and dynamical closed ball $\overline{B}_n(x, \varepsilon)$ are defined respectively as

$$B_n(x, \varepsilon) := \{y \in X \mid d_n(x, y) < \varepsilon\}, \overline{B}_n(x, \varepsilon) := \{y \in X \mid d_n(x, y) \leq \varepsilon\}.$$

A set S is (n, ε) -separated for Z if $S \subset Z$ and $d_n(x, y) > \varepsilon$ for any $x, y \in S$ and $x \neq y$. A set $S \subset Z$ is (n, ε) -spanning for Z if for any $x \in Z$, there exists $y \in S$ such that $d_n(x, y) \leq \varepsilon$.

We have the following definition of entropy for compact set and thus the definition of entropy for a general subset.

Definition 2.1. For $E \subset X$ compact, we have the following Bowen's definition of topological entropy (see [9], c.f. [43]).

$$(2.1) \quad h_{\text{top}}(f, E) = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{\log s_n(E, \varepsilon)}{n}.$$

For a general subset $Y \subset X$, define

$$(2.2) \quad h_{\text{top}}(f, Y) = \sup\{h_{\text{top}}(f, E) : E \subset Y \text{ is compact}\}.$$

Finally, we put $h_{\text{top}}(f) = h_{\text{top}}(f, X)$. Since X is a compact metric space, the definition depends only on the topology on X , i.e. it is independent of the choice of metric defining the same topology on X .

Let $\mu \in \mathcal{M}_f(X)$. Given $\xi = \{A_1, \dots, A_k\}$ a finite measurable partition of X , i.e., a disjoint collection of elements of $\mathfrak{B}(X)$ whose union is X , we define the entropy of ξ by

$$H_\mu(\xi) = - \sum_{i=1}^k \mu(A_i) \log \mu(A_i).$$

The metric entropy of f with respect to ξ is given by

$$h_\mu(f, \xi) = \lim_{n \rightarrow \infty} \frac{1}{n} \log H_\mu\left(\bigvee_{i=0}^{n-1} f^{-i}\xi\right).$$

The metric entropy of f with respect to μ is given by

$$h_\mu(f) = \sup_{\xi} h_\mu(f, \xi),$$

where ξ ranges over all finite measurable partitions of X .

2.2. Bowen Hausdorff Entropy. Bowen also introduced another definition of entropy working for non-compact sets (see [10]). Some author call this notion *Bowen Hausdorff entropy* (e.g. see [23]).

Definition 2.2. For a general subset $E \subseteq X$, let $\mathcal{G}_n(E, \sigma)$ be the collection of all finite or countable covers of E by sets of the form $B_u(x, \sigma)$ with $u \geq n$. We set

$$C(E; t, n, \sigma, f) := \inf_{\mathcal{C} \in \mathcal{G}_n(E, \sigma)} \sum_{B_u(x, \sigma) \in \mathcal{C}} e^{-tu}$$

and

$$C(E; t, \sigma, f) := \lim_{n \rightarrow \infty} C(E; t, n, \sigma, f).$$

Then

$$h_{\text{top}}^B(E; \sigma, f) := \inf\{t : C(E; t, \sigma, f) = 0\} = \sup\{t : C(E; t, \sigma, f) = \infty\}$$

and the Bowen Hausdorff entropy of E is

$$(2.3) \quad h_{\text{top}}^B(f, E) := \lim_{\sigma \rightarrow 0} h_{\text{top}}^B(E; \sigma, f).$$

Finally, we put $h_{\text{top}}^B(f) = h_{\text{top}}^B(f, X)$.

It was proved by Bowen that $h_{top}^B(f) = h_{top}(f)$, however on subsets of X we cannot guarantee this equality. For example, if A is a countable set then $h_{top}^B(f, A) = 0$, while it may happen that $h_{top}(f, A) > 0$. In fact, it can be proved (cf. [32]) that when invertible minimal dynamical system with finite entropy then for any $0 \leq \alpha \leq \beta \leq h_{top}(f)$ there is plenty of compact sets A with $h_{top}^B(f, A) = \alpha$ and $h_{top}(f, A) = \beta$. Then the cases $h_{top}^B(f, A) = h_{top}(f, A)$ are in some sense special.

Let $\mu \in \mathcal{M}(X)$. The *measure-theoretical lower and upper entropies* of μ are defined respectively by

$$\underline{h}_\mu(f) = \int \underline{h}_\mu(f, x) d\mu(x), \quad \overline{h}_\mu(f) = \int \overline{h}_\mu(f, x) d\mu(x),$$

where

$$\begin{aligned} \underline{h}_\mu(f, x) &= \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow +\infty} -\frac{1}{n} \log \mu(B_n(x, \varepsilon)), \\ \overline{h}_\mu(f, x) &= \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow +\infty} -\frac{1}{n} \log \mu(B_n(x, \varepsilon)). \end{aligned}$$

Brin and Katok [14] proved that for any $\mu \in M(X, f)$, $\underline{h}_\mu(f, x) = \overline{h}_\mu(f, x)$ for μ a.e. $x \in X$, and $\int \underline{h}_\mu(f, x) d\mu(x) = h_\mu(f)$. So for $\mu \in \mathcal{M}_f(X)$,

$$\underline{h}_\mu(f) = \overline{h}_\mu(f) = h_\mu(f).$$

From [19, Proposition 1.2] we know that if $E \subseteq X$ is non-empty and compact, then

$$(2.4) \quad h_{top}^B(f, E) = \sup\{\underline{h}_\mu(f) : \mu \in \mathcal{M}(X), \mu(E) = 1\}.$$

Remark that one similar idea to estimate h_{top}^B can be found in [39], called entropy distribution principle.

Proposition 2.1. (*Entropy Distribution Principle*) *Let $f : X \mapsto X$ be a continuous transformation. Let $Z \subseteq X$ be an arbitrary Borel set. Suppose there exists $\varepsilon > 0$ and $s \geq 0$ such that one can find a sequence of Borel probability measures μ_k , a constant $K > 0$, and a limit measure ν of the sequence μ_k satisfying $\nu(Z) > 0$ such that*

$$\limsup_{k \rightarrow \infty} \mu_k(B_n(x, \varepsilon)) \leq K \exp\{-ns\}$$

for sufficiently large n and every ball $B_n(x, \varepsilon)$ which has non-empty intersection with Z . Then $h_{top}^B(Z, \varepsilon, f) \geq s$.

2.3. Packing topological entropy. Let $E \subseteq X$. For $s \geq 0$, $N \in \mathbb{N}$ and $\varepsilon > 0$, define

$$P_{N, \varepsilon}^s(E) = \sup \sum_i \exp(-sn_i),$$

where the supremum is taken over all finite or countable pairwise disjoint families $\{\overline{B}_{n_i}(x_i, \varepsilon)\}$ such that $x_i \in E$, $n_i \geq N$ for all i , where

$$\overline{B}_n(x, \varepsilon) := \{y \in X : d_n(x, y) \leq \varepsilon\}.$$

The quantity $P_{N, \varepsilon}^s(E)$ does not decrease as N, ε decrease, hence the following limits exist:

$$P_\varepsilon^s(E) = \lim_{N \rightarrow \infty} P_{N, \varepsilon}^s(E).$$

Define

$$\mathcal{P}_\varepsilon^s(E) = \inf \left\{ \sum_{i=1}^{\infty} P_\varepsilon^s(E_i) : \bigcup_{i=1}^{\infty} E_i \supseteq E \right\}.$$

Clearly, $\mathcal{P}_\varepsilon^s$ satisfies the following property: if $E \subseteq \bigcup_{i=1}^{\infty} E_i$, then $\mathcal{P}_\varepsilon^s(E) \leq \sum_{i=1}^{\infty} \mathcal{P}_\varepsilon^s(E_i)$.

There exists a critical value of the parameter s , which we will denote by $h_{top}^P(f, E, \varepsilon)$, where $\mathcal{P}_\varepsilon^s(E)$ jumps from ∞ to 0, i.e.

$$\mathcal{P}_\varepsilon^s(E) = \begin{cases} 0, & s > h_{top}^P(f, E, \varepsilon), \\ \infty, & s < h_{top}^P(f, E, \varepsilon). \end{cases}$$

Note that $h_{top}^P(f, E, \varepsilon)$ increases when ε decreases. We call

$$h_{top}^P(f, E) := \lim_{\varepsilon \rightarrow 0} h_{top}^P(f, E, \varepsilon)$$

the *packing topological entropy of f restricted to E* or, simply, the *packing topological entropy of E* , when there is no confusion about f . This quantity is defined in way which resembles the packing dimension. This definition is from [19].

From [19, Proposition 2.1] we know that for any $E \subseteq X$,

$$(2.5) \quad h_{top}^B(f, E) \leq h_{top}^P(f, E) \leq h_{top}(f, E);$$

and if $Z \subseteq X$ is f -invariant and compact, then

$$(2.6) \quad h_{top}^B(f, Z) = h_{top}^P(f, Z) = h_{top}(f, Z).$$

Moreover, from [19, Proposition 1.3] we know that if $E \subseteq X$ is non-empty and compact, then

$$(2.7) \quad h_{top}^P(f, E) = \sup\{\bar{h}_\mu(f) : \mu \in \mathcal{M}(X), \mu(E) = 1\}.$$

2.4. Katok's definition of metric entropy. We use Katok's definition of metric entropy (see [26]). It is defined for ergodic measure and equivalent to the classical one. Let $Z \subset X$. A set S is (n, ε) -separated for Z if $S \subset Z$ and $d_n(x, y) > \varepsilon$ for any $x, y \in S$ and $x \neq y$. A set $S \subset Z$ is (n, ε) -spanning for Z if for any $x \in Z$, there exists $y \in S$ such that $d_n(x, y) \leq \varepsilon$.

Let $\mu \in \mathcal{M}_f^e(X)$. For $\varepsilon > 0, \rho \in (0, 1)$, let $N_l(\varepsilon, \rho)$ be the minimal number of ε -balls $B_l(x, \varepsilon)$ in the d_l -metric, which cover a set $Z \subseteq X$ of measure at least $1 - \rho$. Define

$$h_\mu(f, \varepsilon) = \liminf_{l \rightarrow \infty} \frac{\log N_l(\varepsilon, \rho)}{l}, \quad h'_\mu(f, \varepsilon) = \limsup_{n \rightarrow \infty} \frac{\log N_l(\varepsilon, \rho)}{l}.$$

Then

$$h_\mu(f) = \lim_{\varepsilon \rightarrow 0} h_\mu(f, \varepsilon) = h'_\mu(f, \varepsilon).$$

We need a following estimate for metric entropy.

Proposition 2.2. *Let $\mu \in \mathcal{M}_f^e(X)$ and $\Gamma \subseteq X$ satisfy $\mu(\Gamma) > 0$. For any $\delta > 0, \varsigma > 0, \varepsilon > 0$, there is some positive integer T_* (only dependent on δ) such that for any large $N > 0$, there is $n \geq N$ and (n, ε) -separated set $E_n \subseteq \Gamma$ such that*

- (1) *for any $x \in E_n$, $f^n x \in \Gamma$;*
- (2) *for any $x, y \in E_n$, $d(f^n x, y) < \delta$;*
- (3) *$\frac{1}{n} \log T_* \# E_n \geq h_\mu(f, \varepsilon) - (2 + 2\varsigma + h_\mu(f, \varepsilon))\varsigma$.*

Before proving Proposition 2.2, we need a following lemma.

Lemma 2.3. *Let $\mu \in \mathcal{M}_f^e(X)$ and $\Gamma \subseteq X$ satisfy $\mu(\Gamma) > 0$. Let $\delta > 0$ and ξ be a finite partition of M with $\text{diam } \xi < \delta$ and $\xi > \{\Gamma, X \setminus \Gamma\}$. For $\rho > 0$, define*

$$\Gamma_{s, \rho} = \{x \in \Gamma : f^m(x) \in \xi(x) \text{ for some } m \in [l, (1 + \rho)l], \text{ for } l \geq s\}.$$

Then $\lim_{s \rightarrow +\infty} \mu(\Gamma_{s, \rho}) = \mu(\Gamma)$.

Proof. This can be deduced from Birkhoff Ergodic Theorem. More precisely, let us first prove a following step.

Lemma 2.4. *Let $\mu \in \mathcal{M}_f^e(X)$ and $\Gamma \subseteq X$ satisfy $\mu(\Delta) > 0$. For $\rho > 0$, define*

$$\Delta_{s,\rho} = \{x \in \Delta : f^m(x) \in \Delta \text{ for some } m \in [l, (1+\rho)l], \text{ for } l \geq s\}.$$

Then $\lim_{s \rightarrow +\infty} \mu(\Delta_{s,\rho}) = \mu(\Delta)$.

Proof. By Birkhoff Ergodic Theorem μ a.e. $y \in \Delta$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_\Delta(f^i y) = \mu(\Delta),$$

where $\chi_\Delta(y)$ denotes the characteristic function on Δ . For given $\rho > 0$, take $\varepsilon > 0$ such that $0 < \frac{1+\varepsilon}{1-\varepsilon} < 1 + \frac{3\rho}{4}$. Then for μ a.e. $y \in \Delta$, there is $N_\varepsilon(y) > 0$ such that for any $n \geq N_\varepsilon(y)$,

$$(2.8) \quad 1 - \varepsilon < \frac{1}{\mu(\Delta)} \frac{1}{n} \sum_{i=0}^{n-1} \chi_\Delta(f^i y) < 1 + \varepsilon.$$

Define $B_s = \{y \in \Delta : N_\varepsilon(y) \leq s\}$, then $\lim_{s \rightarrow +\infty} \mu(B_s) = \mu(\Delta)$. Take $s \geq \frac{4}{\rho}$, we claim $B_s \subseteq \Delta_{s,\rho}$. By contradiction, there is $x \in B_s$ and $l \geq s$ such that

$$f^m(x) \in X \setminus \Delta \text{ for all } m \in [l, (1+\rho)l].$$

Then by the right estimate of (2.8) for $n = l$

$$\begin{aligned} \frac{1}{\mu(\Delta)} \frac{1}{[(1+\rho)l]} \sum_{i=0}^{[(1+\rho)l]-1} \chi_\Delta(f^i y) &= \frac{1}{\mu(\Delta)} \frac{1}{[(1+\rho)l]} \sum_{i=0}^{l-1} \chi_\Delta(f^i y) < \frac{1}{[(1+\rho)l]} l(1+\varepsilon) \\ &\leq \frac{l(1+\varepsilon)}{(1+\rho)l-1} = \frac{(1+\varepsilon)}{(1+\rho)-\frac{1}{l}} \leq \frac{(1+\varepsilon)}{(1+\rho)-\frac{\rho}{4}} < 1 - \varepsilon. \end{aligned}$$

where $[a]$ denotes the integer part of a . This contradicts the left estimate in (2.8) for $n = [(1+\rho)l]$. \square

Now we continue to prove Lemma 2.3. Let $\xi = \{C_1, C_2, \dots, C_k\}$ ($k \geq 1$) and recall that $\xi > \{\Gamma, X \setminus \Gamma\}$. Consider $\Delta = C_i$ and by Lemma 2.4 taking corresponding $(C_i)_{s,\rho}$,

$$\lim_{s \rightarrow +\infty} \mu(\Gamma_{s,\rho}) = \lim_{s \rightarrow +\infty} \mu(\cup_{i=1}^k \Gamma \cap (C_i)_{s,\rho}) = \mu(\cup_{i=1}^k \Gamma \cap C_i) = \mu(\Delta).$$

\square

Proof of Proposition 2.2. Take $\rho \in (0, 1)$ such that $1 - \rho < \mu(\Gamma)$. Since X is compact metric space, we can take $\xi = \{\xi_1, \xi_2, \dots, \xi_q\}$ to be a finite partition of M with $\text{diam } \xi < \frac{1}{2}\delta$ and $\xi > \{\Gamma, X \setminus \Gamma\}$. Fix a positive integer $T_* \geq q = \#\xi$. Let $\Gamma_{s,\varsigma}$ be same as in Lemma 2.3. Then $\mu(\Gamma_{s,\varsigma}) \rightarrow \mu(\Gamma)$ ($as s \rightarrow +\infty$) and thus we can take sufficiently large s such that $\mu(\Gamma_{s,\varsigma}) > 1 - \rho$. Let $D_l \subseteq \Gamma_{s,\varsigma}$ be an (l, ε) -separated set of maximal cardinality; in other words, the cover by ε -balls in the d_l -metric centered at points in D_l is a cover of $\Gamma_{s,\varsigma}$. Then we have

$$h_\mu(f, \varepsilon) \leq \liminf_{l \rightarrow +\infty} \frac{1}{l} \log \# D_l.$$

So for given N there exists $L > N$ such that,

$$h_\mu(f, \varepsilon) - \varsigma < \frac{1}{L} \log \# D_L,$$

and

$$L_\varsigma < e^{L\varsigma}.$$

Let

$$G_l = \{x \in D_L : f^l(x) \in \xi(x)\}, \quad l = L, L+1, \dots, [(1+\varsigma)L].$$

By definition of $\Gamma_{L,\varsigma}$, from $D_L \subseteq \Gamma_{L,\varsigma}$ we can take $n \geq N$ with $n \in [L, (1+\varsigma)L]$ such that

$$\begin{aligned} \#G_n &\geq \frac{1}{[(1+\varsigma)L] - L} \sum_{l=L}^{[(1+\varsigma)L]-1} \#G_l \geq \frac{1}{[(1+\varsigma)L] - L} \#D_L \geq \frac{1}{\varsigma L} \#D_L \\ &\geq e^{-L\varsigma + L(h_\mu(f,\varepsilon) - \varsigma)} = e^{(-2\varsigma + h_\mu(f,\varepsilon))L} = e^{(-2\varsigma + h_\mu(f,\varepsilon))n} e^{(-2\varsigma + h_\mu(f,\varepsilon))(L-n)} \\ &\geq e^{(-2\varsigma + h_\mu(f,\varepsilon))n} e^{-|(-2\varsigma + h_\mu(f,\varepsilon))(L-n)|} \geq e^{(-2\varsigma + h_\mu(f,\varepsilon))n} e^{-(2\varsigma + h_\mu(f,\varepsilon))\varsigma L} \\ &\geq e^{(-2\varsigma + h_\mu(f,\varepsilon))n} e^{-(2\varsigma + h_\mu(f,\varepsilon))\varsigma n} = e^{nh_\mu(f,\varepsilon) - (2+2\varsigma + h_\mu(f,\varepsilon))\varsigma n}. \end{aligned}$$

Let $F_j := G_n \cap \xi_j$, then we can take some F_{j_0} , denoted by E_n , such that

$$\#E_n = \max_{j=1}^q \#G_n \cap \xi_j \geq \frac{1}{q} \#G_n \geq \frac{1}{T_*} e^{nh_\mu(f,\varepsilon) - (2+2\varsigma + h_\mu(f,\varepsilon))\varsigma n}.$$

This gives the estimate (3). (1) is from the definition of G_n and $G_n \supseteq E_n$. By the choice ξ with $\text{diam}(\xi) < \frac{1}{2}\delta$ and $E_n \subseteq \xi_{j_0}$, for any $x, y \in E_n$, we have

$$d(f^n x, y) \leq d(f^n x, x) + d(x, y) \leq 2\text{diam}(\xi_{j_0}) < \delta.$$

This is item (2). Now we complete the proof. \square

2.5. (Exponential) Shadowing. We begin with the classical shadowing property. Let (X, d) be a compact metric space and $f : X \mapsto X$ be a homeomorphism. A sequence $\{x_n\}_{n \in \mathbb{Z}} \subset M$ is called a δ -pseudo-orbit of f if

$$d(f(x_n), x_{n+1}) < \delta \text{ for any } n \in \mathbb{Z}.$$

Moreover, a δ -pseudo-orbit $\{x_n\}_{n \in \mathbb{Z}}$ is τ -shadowed by an orbit of $y \in M$ if

$$d(f^n(y), x_n) < \tau \text{ for any } n \in \mathbb{Z}.$$

Finally, we say $f : M \rightarrow M$ satisfies shadowing property if for any $\tau > 0$, there exists $\delta > 0$ such that any δ -pseudo-orbit is τ -shadowed by some orbit.

Given $x \in M$ and $n \in \mathbb{N}$, let

$$\{x, n\} := \{f^j(x) \mid j = 0, 1, \dots, n-1\}.$$

In other words, $\{x, n\}$ represents the orbit segment from x with length n . For a sequence of points $\{x_i\}_{i=-\infty}^{+\infty}$ in M and a sequence of positive integers $\{n_i\}_{i=-\infty}^{+\infty}$, we call $\{x_i, n_i\}_{i=-\infty}^{+\infty}$ a δ -pseudo-orbit, if

$$d(f^{n_i}(x_i), x_{i+1}) < \delta$$

for all i .

Given $\lambda > 0$ and $\tau > 0$, we call a point $x \in M$ a (τ, λ) -shadowing point for a pseudo-orbit $\{x_i, n_i\}_{i=-\infty}^{+\infty}$, if

$$d(f^{c_i+j}(x), f^j(x_i)) < \tau \cdot e^{-\min\{j, n_i-j\}\lambda},$$

$\forall j = 0, 1, 2, \dots, n_i - 1$ and $\forall i \in \mathbb{Z}$, where c_i is defined as

$$(2.9) \quad c_i = \begin{cases} 0, & \text{for } i = 0 \\ \sum_{j=0}^{i-1} n_j, & \text{for } i > 0 \\ -\sum_{j=i}^{-1} n_j, & \text{for } i < 0. \end{cases}$$

Now we start to introduce *exponential shadowing* property.

Definition 2.5. Let $\lambda > 0$. f is called to have λ -exponential shadowing property, if the following holds: for any $\tau > 0$ there exists $\delta > 0$ such that for any δ -pseudo-orbit $\{x_i, n_i\}_{i=-\infty}^{+\infty}$, there is a (τ, λ) -shadowing point $x \in X$ for $\{x_i, n_i\}_{i=-\infty}^{+\infty}$.

Moreover, we say f to have periodic λ -exponential shadowing property, if it has λ -exponential shadowing property and furthermore, if $\{x_i, n_i\}_{i=-\infty}^{+\infty}$ in above definition is periodic, i.e., there exists an integer $m > 0$ such that $x_{i+m} = x_i$ and $n_{i+m} = n_i$ for all i , then the shadowing point x is periodic.

Remark 2.6. If a homeomorphism f is topologically conjugated to a homeomorphism g satisfying (periodic) β -exponential shadowing property for some $\beta > 0$, and the inverse conjugation is γ -Hölder continuous, then it is not difficult to see that f has (periodic) $\beta\gamma$ -exponential shadowing property.

Remark that if above $\{x_i, n_i\}_{i=-\infty}^{+\infty}$ is periodic with $m = 1$, it is in fact the concept of *exponential closing* introduced in [16, 25]. Exponential shadowing (or closing) plays important role in the estimate of Lyapunov exponents, for example, see [25, 44, 31, 38].

For convenience, we say the orbit segments $x, fx, \dots, f^n x$ and $y, fy, \dots, f^n y$ are exponentially τ close with exponent λ , meaning that

$$d(f^i(x), f^i(y)) < \tau e^{-\lambda \min\{i, n-i\}}, 0 \leq i \leq n-1.$$

We will show that any hyperbolic set has exponential shadowing.

Proposition 2.7. *Let $f : M \rightarrow M$ be a C^1 diffeomorphism, with an invariant hyperbolic locally maximal closed set Λ . Then $f|_\Lambda$ has exponential shadowing property.*

Before proving Proposition 2.7, let us recall Katok's shadowing lemma in the case of C^1 hyperbolic system (c.f. [16], firstly proved in Chap. 5 [37] for $C^{1+\alpha}$ non-uniformly hyperbolic case). Let $(\delta_k)_{k=1}^{+\infty}$ be a sequence of positive real numbers. Let $(x_n)_{n=-\infty}^{+\infty}$ be a sequence of points in X for which there exists a sequence $(s_n)_{n=-\infty}^{+\infty}$ of positive integers satisfying:

- (i) $|s_n - s_{n-1}| \leq 1, \forall n \in \mathbb{Z}$;
- (ii) $d(fx_n, x_{n+1}) \leq \delta_{s_n}, \forall n \in \mathbb{Z}$;

then we call $(x_n)_{n=-\infty}^{+\infty}$ a $(\delta_k)_{k=1}^{+\infty}$ pseudo-orbit. Given $\tau > 0$, a point $x \in M$ is a (τ, λ) -shadowing point for the $(\delta_k)_{k=1}^{+\infty}$ pseudo-orbit if $d(f^n x, x_n) \leq \tau \lambda_{s_n}, \forall n \in \mathbb{Z}$, where $\lambda_k = \lambda_0 e^{-\lambda k}$ and λ_0 is a constant only dependent on the system of f .

Lemma 2.8. (*Katok's Shadowing lemma*) *Let $f : M \rightarrow M$ be a C^1 diffeomorphism, with an invariant hyperbolic closed set Λ . Then there exists $\theta_0 > 0$ and $\lambda_* > 0$ such that for any $\tau \in (0, \theta_0), \lambda \in (0, \lambda_*)$, there exists a sequence $(\delta_k)_{k=1}^{+\infty}$ such that for any $(\delta_k)_{k=1}^{+\infty}$ pseudo-orbit contained in Λ there exists a unique (τ, λ) -shadowing point in M .*

Proof of Proposition 2.7. Let $\theta_0 > 0$ and $\lambda_* > 0$ same as in Lemma 2.8. Fix $\lambda \in (0, \lambda_*)$ and define corresponding sequence $\lambda_k = \lambda_0 e^{-\lambda k}$ where λ_0 is a constant only dependent on the system of f . For $\tau > 0$, by local maximal property of Λ , we can take τ_1 such that $0 < \tau_1 < \min\{\tau, \theta_0\}$ and any point y with $\sup_{z \in Orb(y)} d(z, \Lambda) < \tau_1 \lambda_0$ should be in Λ . For τ_1 and λ , using Lemma 2.8, there exists a sequence $(\delta_k)_{k=1}^{+\infty}$ such that for any $(\delta_k)_{k=1}^{+\infty}$ pseudo-orbit contained in Λ there exists a unique (τ_1, λ) -shadowing point in M . By the choice of τ_1 , such showing point should be in Λ .

Fix large k such that $\lambda_k = \lambda_0 e^{-\lambda k} < 1$. Take $\delta = \delta_k$. For any δ -pseudo-orbit $\{x_i, n_i\}_{i=-\infty}^{+\infty}$ contained in Λ , define $y_j = f^{j-c_i}(x_i)$ and define

$$s_j = k + \min\{j - c_i, c_{i+1} - j\},$$

for $c_i \leq j \leq c_{i+1} - 1, \forall i \in \mathbb{Z}$. Then $|s_j - s_{j-1}| \leq 1, \forall j \in \mathbb{Z}$ and for $\forall i \in \mathbb{Z}$,

$$d(f(y_j), y_{j+1}) = 0 < \delta_{s_j},$$

for j with $c_i \leq j \leq c_{i+1} - 2$ and for $j = c_{i+1} - 1$,

$$d(f(y_j), y_{j+1}) = d(f(y_{c_{i+1}-1}), y_{c_{i+1}}) = d(f^{n_i}(x_i), x_{i+1}) < \delta = \delta_k = \delta_{s_j}.$$

This implies that $\{x_i, n_i\}_{i=-\infty}^{+\infty}$ is a $(\delta_k)_{k=1}^{+\infty}$ pseudo-orbit contained in Λ . So there is a unique (τ_1, λ) -shadowing point y in Λ . More precisely, for $c_i \leq j \leq c_{i+1} - 1$, $i \in \mathbb{Z}$,

$$d(f^j(y), y_j) < \tau_1 \lambda_{s_j} < \tau \lambda_0 e^{-\lambda s_j} < \tau e^{-\lambda(s_j - k)} = \tau e^{-\lambda \min\{j - c_i, c_{i+1} - j\}}.$$

In other words,

$$d(f^{c_i+j}(x), f^j(x_i)) < \tau \cdot e^{-\min\{j, n_i - j\}\lambda},$$

$\forall j = 0, 1, 2, \dots, n_i - 1$ and $\forall i \in \mathbb{Z}$. □

Remark 2.9. One also can use another way to prove Proposition 2.7, since it is not difficult to see that

Shadowing + Local product structure \Rightarrow Exponential shadowing,

and it is known that every hyperbolic set has local product structure and every locally maximal hyperbolic set satisfies shadowing property whose shadowing point is contained in the given set.

Example 2.10. Non-hyperbolic systems with exponential shadowing: From [22] we know that non-hyperbolic diffeomorphism f with C^{1+Lip} smoothness, conjugated to a transitive Anosov diffeomorphism, exists even the conjugation and its inverse is Hölder continuous. This example is non-hyperbolic but satisfies exponential shadowing property.

2.6. Semi-Continuity of MLE w.r.t. measures.

Proposition 2.11. *Let (X, d) be a compact metric space and $f : X \mapsto X$ be a homeomorphism. Let $A : X \rightarrow GL(m, \mathbb{R})$ be a continuous matrix function. Then $\chi_{max}(A, \mu)$ is upper semi-continuous with respect to $\mu \in \mathcal{M}_f(X)$.*

Proof. Observe that

$$\begin{aligned} \chi_{max}(A, \mu) &= \int \chi_{max}(A, x) d\mu \\ &= \lim_{n \rightarrow +\infty} \int \frac{1}{n} \log \|A(x, n)\| d\mu = \inf_{n \geq 1} \int \frac{1}{n} \log \|A(x, n)\| d\mu. \end{aligned}$$

The upper semi-continuity follows. □

Now let's recall the entropy-dense property of Theorem 2.1 in [34] (or see [35], also see [18] for similar discussion). Roughly speaking, any invariant probability measure μ is the limit of a sequence of ergodic measures $\{\mu_n\}_{n=1}^{\infty}$ in weak* topology such that the entropy of μ is the limit of the entropies of μ_n . We say T has *entropy-dense* property, if for any $\nu \in \mathcal{M}_f(X)$, any neighborhood $G \subseteq \mathcal{M}(X)$ of ν and any $h_* < h_\nu(f)$, there exists an ergodic measure $\mu \in G \cap \mathcal{M}_f(X)$ such that $h_\mu(f) > h_*$.

Proposition 2.12. *Let (X, d) be a compact metric space and $f : X \mapsto X$ be a homeomorphism. Let $A : X \rightarrow GL(m, \mathbb{R})$ be a continuous matrix function. If there are two ergodic measures with different MLE w.r.t. A , then*

(1)

$$h_{top}(f) = \sup_{\mu, \nu \in \mathcal{M}_f^e(X)} \{\max\{h_\mu(f), h_\nu(f)\} \mid \chi_{max}(\mu, A) > \chi_{max}(\nu, A)\};$$

(2) *If f satisfies entropy-dense property, then*

$$\sup_{\mu \in \mathcal{M}_f^e(X)} \{h_\mu(f) \mid \chi_{max}(\mu, A) > \inf_{\nu \in \mathcal{M}_f(X)} \chi_{max}(\nu, A)\}$$

$$= \sup_{\mu, \nu \in \mathcal{M}_f^e(X)} \{\min\{h_\mu(f), h_\nu(f)\} \mid \chi_{\max}(\mu, A) > \chi_{\max}(\nu, A)\}.$$

If further $\chi_{\max}(A, \mu)$ is lower semi-continuous with respect to $\mu \in \mathcal{M}_f(X)$, then they are equal to the full entropy, $h_{\text{top}}(f)$.

Proof. (1) Assume that

$$\sup_{\nu \in \mathcal{M}_f^e(X)} \chi_{\max}(\nu, A) > \inf_{\nu \in \mathcal{M}_f^e(X)} \chi_{\max}(\nu, A).$$

We only need to prove for any $\gamma > 0$,

$$h_{\text{top}}(f) \leq \sup_{\mu, \nu \in \mathcal{M}_f^e(X)} \{\max\{h_\mu(f), h_\nu(f)\} \mid \chi_{\max}(\mu, A) > \chi_{\max}(\nu, A)\} + \gamma.$$

By classical variational principle (see Chapter 7 in [43]), there is an ergodic measure ω such that $h_\omega(f) \geq h_{\text{top}}(f) - \gamma$. By assumption, take another $\varpi \in \mathcal{M}_f^e(X)$ such that $\chi_{\max}(\omega, A) \neq \chi_{\max}(\varpi, A)$. Then

$$\begin{aligned} h_{\text{top}}(f) &\leq h_\omega(f) + \gamma \leq \max\{h_\omega(f), h_\varpi(f)\} + \gamma \\ &\leq \sup_{\mu, \nu \in \mathcal{M}_f^e(X)} \{\max\{h_\mu(f), h_\nu(f)\} \mid \chi_{\max}(\mu, A) > \chi_{\max}(\nu, A)\} + \gamma. \end{aligned}$$

(2) The part of \geq is trivial. So we only need to prove the part of \leq . Suppose $\mu \in \mathcal{M}_f^e(X)$ satisfying $\chi_{\max}(\mu, A) > \inf_{\omega \in \mathcal{M}_f(X)} \chi_{\max}(\omega, A)$. Then there is $\omega \in \mathcal{M}_f(X)$ such that $\chi_{\max}(\mu, A) > \chi_{\max}(\omega, A)$. For $\gamma > 0$, take $\theta \in (0, 1)$ close to 1 enough such that $\varpi = \theta\mu + (1-\theta)\omega$ has entropy larger than $h_\mu(f) - \gamma$. Remark that $\chi_{\max}(\varpi, A) < \chi_{\max}(\mu, A)$ and take $\zeta < \chi_{\max}(\mu, A) - \chi_{\max}(\varpi, A)$. By Proposition 2.11 and the assumption of entropy-dense property, there is $\nu \in \mathcal{M}_f^e(X)$ close ϖ enough such that $h_\nu(f) > h_\varpi(f) - \gamma > h_\mu(f) - 2\gamma$ and

$$\chi_{\max}(\nu, A) < \chi_{\max}(\varpi, A) + \zeta < \chi_{\max}(\mu, A).$$

Thus

$$\begin{aligned} &\sup_{\mu', \nu' \in \mathcal{M}_f^e(X)} \{\min\{h_{\mu'}(f), h_{\nu'}(f)\} \mid \chi_{\max}(\mu', A) > \chi_{\max}(\nu', A)\} \\ &\geq \min\{h_\mu(f), h_\nu(f)\} > h_\mu(f) - 2\gamma. \end{aligned}$$

If further $\chi_{\max}(A, \mu)$ is lower semi-continuous with respect to $\mu \in \mathcal{M}_f(X)$, then we are going to prove full entropy, $h_{\text{top}}(f)$. We only need to prove for any $\gamma > 0$,

$$h_{\text{top}}(f) \leq \sup_{\mu \in \mathcal{M}_f^e(X)} \{h_\mu(f) \mid \chi_{\max}(\mu, A) > \inf_{\nu \in \mathcal{M}_f(X)} \chi_{\max}(\nu, A)\} + 2\gamma.$$

By classical variational principle (see Chapter 7 in [43]), there is an ergodic measure ω' such that $h_{\omega'}(f) \geq h_{\text{top}}(f) - \gamma$. If $\chi_{\max}(\omega', A) > \inf_{\nu \in \mathcal{M}_f(X)} \chi_{\max}(\nu, A)$, then

$$h_{\text{top}}(f) \leq h_{\omega'}(f) + \gamma \leq \sup_{\mu \in \mathcal{M}_f^e(X)} \{h_\mu(f) \mid \chi_{\max}(\mu, A) > \inf_{\nu \in \mathcal{M}_f(X)} \chi_{\max}(\nu, A)\} + \gamma.$$

Otherwise, $\chi_{\max}(\omega', A) = \inf_{\nu \in \mathcal{M}_f(X)} \chi_{\max}(\nu, A)$. By assumption, take another $\varpi' \in \mathcal{M}_f^e(X)$ such that $\chi_{\max}(\omega', A) \neq \chi_{\max}(\varpi', A)$ and then

$$\chi_{\max}(\varpi', A) > \inf_{\nu \in \mathcal{M}_f(X)} \chi_{\max}(\nu, A) = \chi_{\max}(\omega', A).$$

By assumption of lower semi-continuity and entropy-dense, there is $\mu' \in \mathcal{M}_f^e(X)$ close ϖ' enough such that $h_{\mu'}(f) > h_{\varpi'}(f) - \gamma > h_{\text{top}}(f) - 2\gamma$ and

$$\chi_{\max}(\mu', A) > \chi_{\max}(\omega', A) = \inf_{\nu \in \mathcal{M}_f(X)} \chi_{\max}(\nu, A).$$

Then

$$h_{top}(f) \leq h_{\mu'}(f) + 2\gamma \leq \sup_{\mu \in \mathcal{M}_f^e(X)} \{h_{\mu}(f) | \chi_{max}(\mu, A) > \inf_{\nu \in \mathcal{M}_f(X)} \chi_{max}(\nu, A)\} + 2\gamma.$$

□

2.7. Lyapunov Exponents and Lyapunov Metric. Suppose $f : X \rightarrow X$ to be an invertible map on a compact metric space X and $A : X \rightarrow GL(m, \mathbb{R})$ to be a continuous matrix function.

Definition 2.13. For any $x \in X$ and any $0 \neq v \in \mathbb{R}^m$, define the Lyapunov exponent of vector v at x ,

$$\lambda(A, x, v) := \lim_{n \rightarrow +\infty} \frac{1}{n} \log \|A(x, n)v\|,$$

if the limit exists. We say x to be (forward) *Lyapunov-regular* for A , if $\lambda(A, x, v)$ exists for all vector $v \in \mathbb{R}^m \setminus \{0\}$. Otherwise, x is called to be *Lyapunov-irregular* for A . Let $LI(A, f)$ denote the space of all Lyapunov-irregular points for A .

By Oseledec's Multiplicative Ergodic theorem, for any invariant μ and μ a.e. x , the Lyapunov exponent $\lambda(A, x, v)$ exists at x for all vectors $v \in \mathbb{R}^m$.

Oseledec Multiplicative Ergodic Theorem [5, Theorem 3.4.4]: Let f be an invertible ergodic measure-preserving transformation of a Lebesgue probability measure space (X, μ) . Let A be a measurable cocycle whose generator satisfies $\log \|A^{\pm}(x)\| \in L^1(X, \mu)$. Then there exist numbers

$$\chi_1 < \chi_2 < \cdots < \chi_l,$$

an f -invariant set \mathcal{R}^{μ} with $\mu(\mathcal{R}^{\mu}) = 1$, and an A -invariant Lyapunov decomposition of \mathbb{R}^m for $x \in \mathcal{R}^{\mu}$,

$$\mathbb{R}_x^m = E_{\chi_1}(x) \oplus E_{\chi_2}(x) \oplus \cdots \oplus E_{\chi_l}(x)$$

with $\dim E_{\chi_i}(x) = m_i$, such that for any $i = 1, \dots, l$ and any $0 \neq v \in E_{\chi_i}(x)$ one has

$$\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|A(x, n)v\| = \chi_i$$

and

$$\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \det A(x, n) = \sum_{i=1}^l m_i \chi_i.$$

Definition 2.14. The numbers $\chi_1, \chi_2, \dots, \chi_l$ are called the *Lyapunov exponents* of measure μ for cocycle A and the dimension m_i of the space $E_{\chi_i}(x)$ is called the *multiplicity* of the exponent χ_i . The collection of pairs

$$Sp(\mu, A) = \{(\chi_i, m_i) : 1 \leq i \leq l\}$$

is the *Lyapunov spectrum* of measure μ .

Remark that for any ergodic measure μ , all the points in the set \mathcal{R}^{μ} are Lyapunov-regular.

We denote the standard scalar product in \mathbb{R}^m by $\langle \cdot, \cdot \rangle$. For a fixed $\epsilon > 0$ and a regular point x we introduce the ϵ -Lyapunov scalar product (or metric) $\langle \cdot, \cdot \rangle_{x, \epsilon}$ in \mathbb{R}^m as follows. For $u \in E_{\chi_i}(x)$, $v \in E_{\chi_j}(x)$, $i \neq j$ we define $\langle \cdot, \cdot \rangle_{x, \epsilon} = 0$. For $i = 1, \dots, l$ and $u, v \in E_{\chi_i}(x)$, we define

$$\langle \cdot, \cdot \rangle_{x, \epsilon} = m \sum_{n \in \mathbb{Z}} \langle A(x, n)u, A(x, n)v \rangle \exp(-2\chi_i n - \epsilon|n|).$$

Note that the series converges exponentially for any regular x . The constant m in front of the conventional formula is introduced for more convenient comparison with

the standard scalar product. Usually, ϵ will be fixed and we will denote $\langle \cdot, \cdot \rangle_{x,\epsilon}$ simply by $\langle \cdot, \cdot \rangle_x$ and call it the *Lyapunov scalar product*. The norm generated by this scalar product is called the *Lyapunov norm* and is denoted by $\|\cdot\|_{x,\epsilon}$ or $\|\cdot\|_x$.

We summarize below some important properties of the Lyapunov scalar product and norm; for more details see [5, §3.5.1-3.5.3]. A direct calculation shows [5, Theorem 3.5.5] that for any regular x and any $u \in E_{\chi_i}(x)$, $\forall n \in \mathbb{Z}$,

$$(2.10) \quad \exp(n\chi_i - \epsilon|n|)\|u\|_{x,\epsilon} \leq \|A(x, n)u\|_{f^n x, \epsilon} \leq \exp(n\chi_i + \epsilon|n|)\|u\|_{x,\epsilon},$$

$$(2.11) \quad \exp(n\chi - \epsilon|n|) \leq \|A(x, n)u\|_{f^n x \leftarrow x} \leq \exp(n\chi + \epsilon|n|),$$

where $\chi = \chi_l$ is the maximal Lyapunov exponent and $\|\cdot\|_{f^n x \leftarrow x}$ is the operator norm with respect to the Lyapunov norms. It is defined for any matrix A and any regular points x, y as follows:

$$\|A\|_{y \leftarrow x} = \sup\{\|Au\|_{y,\epsilon} \cdot \|u\|_{x,\epsilon}^{-1} : 0 \neq u \in \mathbb{R}^m\}.$$

We emphasize that, for any given $\epsilon > 0$, Lyapunov scalar product and Lyapunov norm are defined only for regular points with respect to the given measure. They depend only measurably on the point even if the cocycle is Hölder. Therefore, comparison with the standard norm becomes important. The uniform lower bound follow easily from the definition:

$$\|u\|_{x,\epsilon} \geq \|u\|.$$

The upper bound is not uniform, but it changes slowly along the regular orbits ([5, Prop. 3.5.8]): there exists a measurable function $K_\epsilon(x)$ defined on the set of regular points \mathcal{R}^μ such that

$$(2.12) \quad \|u\| \leq \|u\|_{x,\epsilon} \leq K_\epsilon(x)\|u\| \quad \forall x \in \mathcal{R}^\mu, \forall u \in \mathbb{R}^m$$

$$(2.13) \quad K_\epsilon(x)e^{-\epsilon n} \leq K_\epsilon(f^n x) \leq K_\epsilon(x)e^{\epsilon n} \quad \forall x \in \mathcal{R}^\mu, \forall n \in \mathbb{Z}.$$

These estimates are obtained in [5] using the fact that $\|u\|_{x,\epsilon}$ is *tempered*, but they can also be checked directly using the definition of $\|u\|_{x,\epsilon}$ on each Lyapunov space and noting that angles between the spaces change slowly.

For any matrix A and any regular points x, y , inequalities (2.12) and (2.13) yield

$$(2.14) \quad K_\epsilon(x)^{-1}\|A\| \leq \|A\|_{y \leftarrow x} \leq K_\epsilon(y)\|A\|.$$

When ϵ is fixed we will usually omit it and write $K(x) = K_\epsilon(x)$. For any $l > 1$ we also define the following sets of regular points

$$(2.15) \quad \mathcal{R}_{\epsilon,l}^\mu = \{x \in \mathcal{R}^\mu : K_\epsilon(x) \leq l\}.$$

Note that $\mu(\mathcal{R}_{\epsilon,l}^\mu) \rightarrow 1$ as $l \rightarrow \infty$. Without loss of generality, we can assume that the set $\mathcal{R}_{\epsilon,l}^\mu$ is compact and that Lyapunov splitting and Lyapunov scalar product are continuous on $\mathcal{R}_{\epsilon,l}^\mu$. Indeed, by Luzin's theorem we can always find a subset of $\mathcal{R}_{\epsilon,l}^\mu$ satisfying these properties with arbitrarily small loss of measure (for standard Pesin sets these properties are automatically satisfied).

2.8. Estimate of the norm of Hölder cocycles. Let us recall some useful lemmas as follows. Firstly let us recall a general estimate of the norm of A along any orbit segment close to a regular one[25].

Lemma 2.15. [25, Lemma 3.1] *Let A be an α -Hölder cocycle ($\alpha > 0$) over a continuous map f of a compact metric space X and let μ be an ergodic measure for f with the largest Lyapunov exponent χ . Then for any positive λ and ϵ satisfying $\lambda > \epsilon/\alpha$ there exists $c > 0$ such that for any $n \in \mathbb{N}$, any regular point x with both x*

and $f^n x$ in $\mathcal{R}_{\epsilon, l}^\mu$, and any point $y \in X$ such that the orbit segments $x, fx, \dots, f^n x$ and $y, fy, \dots, f^n(y)$ are exponentially τ close with exponent λ for some $\tau > 0$ we have

$$(2.16) \quad \|A(y, n)\|_{f^n x \leftarrow x} \leq e^{cl\tau^\alpha} e^{n(\chi+\epsilon)} \leq e^{2n\epsilon+cl\tau^\alpha} \|A(x, n)\|_{f^n x \leftarrow x}$$

and

$$(2.17) \quad \|A(y, n)\| \leq l^2 e^{cl\tau^\alpha} e^{n(\chi+\epsilon)} \leq l^2 e^{2n\epsilon+cl\tau^\alpha} \|A(x, n)\|.$$

The constant c depends only on the cocycle A and on the number $(\alpha\lambda - \epsilon)$.

Lemma 2.16. *Let A be an α -Hölder cocycle ($\alpha > 0$) over a continuous map f of a compact metric space X and let μ be an ergodic measure for f with the largest Lyapunov exponent χ . Then for any positive λ and ϵ satisfying $\lambda > \epsilon/\alpha$ there exists $\tau > 0$ such that for any $n \in \mathbb{N}$, any regular point x with both x and $f^n x$ in $\mathcal{R}_{\epsilon, l}^\mu$, and any point $y \in X$ such that the orbit segments $x, fx, \dots, f^n x$ and $y, fy, \dots, f^n(y)$ are exponentially τ close with exponent λ for some $\tau > 0$ we have*

$$(2.18) \quad \|A(y, n)\| \leq l^2 e^l e^{n(\chi+\epsilon)} \leq l^2 e^l e^{2n\epsilon} \|A(x, n)\|.$$

Proof. For Lemma 2.15, let $\tau > 0$ small enough such that

$$c\tau^\alpha < 1.$$

Then the estimate (2.18) is obvious from Lemma 2.15. \square

Another lemma is to estimate the growth of vectors in a certain cone $K \subseteq \mathbb{R}^m$ invariant under $A(x, n)$ [25]. Let x be a point in $\mathcal{R}_{\epsilon, l}^\mu$ and $y \in X$ be a point such that the orbit segments $x, fx, \dots, f^n x$ and $y, fy, \dots, f^n y$ are exponentially τ close with exponent λ . We denote $x_i = f^i x$ and $y_i = f^i y$, $i = 0, 1, \dots, n$. For each i we have orthogonal splitting $\mathbb{R}^m = E_i \oplus F_i$, where E_i is the Lyapunov space at x_i corresponding to the largest Lyapunov exponent χ and F_i is the direct sum of all other Lyapunov spaces at x_i corresponding to the Lyapunov exponents less than χ . For any vector $u \in \mathbb{R}^m$ we denote by $u = u' + u^\perp$ the corresponding splitting with $u' \in E_i$ and $u^\perp \in F_i$; the choice of i will be clear from the context. To simplify notation, we write $\|\cdot\|_i$ for the Lyapunov norm at x_i . For each $i = 0, 1, \dots, n$ we consider cones

$$K_i = \{u \in \mathbb{R}^m : \|u^\perp\|_i \leq \|u'\|_i\} \quad \text{and} \quad K_i^\eta = \{u \in \mathbb{R}^m : \|u^\perp\|_i \leq (1 - \eta)\|u'\|_i\}$$

with $\eta > 0$. Remark that

$$(2.19) \quad \|u\|_i \geq \|u'\|_i \geq \frac{1}{\sqrt{2}} \|u\|_i.$$

We will consider the case when χ is not the only Lyapunov exponent of A with respect to μ . Otherwise $F_i = \{0\}$, $K_i^\eta = K_i = \mathbb{R}^m$, and the argument becomes simpler. Recall that $\epsilon < \epsilon_0 = \min\{\lambda\alpha, (\chi - \nu)/2\}$, where $\nu < \chi$ is the second largest Lyapunov exponent of A with respect to μ .

Lemma 2.17. [25, Lemma 3.3] *In the notation above, for any regular set $\mathcal{R}_{\epsilon, l}^\mu$, there exist $\eta, \tau > 0$ such that if $x, f^n x \in \mathcal{R}_{\epsilon, l}^\mu$ and the orbit segments $x, fx, \dots, f^n x$ and $y, fy, \dots, f^n(y)$ are exponentially τ close with exponent λ , then for every $i = 0, 1, \dots, n-1$ we have $A(y_i)(K_i) \subseteq K_i^\eta$ and $\|(A(y_i)u)'\|_{i+1} \geq e^{\chi-2\epsilon} \|u'\|_i$ for any $u \in K_i$.*

3. PROOF

3.1. ML-irregular set and Topological Entropy.

Theorem 3.1. *Let (X, d) be a compact metric space and $f : X \mapsto X$ be a topologically mixing homeomorphism with exponential shadowing property and let $A : M \rightarrow GL(m, \mathbb{R})$ be a Hölder continuous matrix function. Then either*

- (1) *all ergodic measures have same MLE w.r.t. A ; or*
- (2) *ML-irregular set $MLI(A, f)$ has full topological entropy and has full packing topological entropy and moreover, its Bowen Hausdorff entropy can be estimated from below by*

$$\sup_{\mu \in \mathcal{M}_f^e(X)} \{h_\mu(f) \mid \mu \text{ is not Lyapunov minimizing for } A\}.$$

Proof. Let A be α -Hölder continuous and let $C := \max_{x \in X} \|A^\pm(x)\|$. Firstly, let us to show the estimate of entropy by metric entropy of two measures with different Lyapunov exponents. Let μ_1 and μ_2 be two ergodic measures such that

$$\int \chi_{\max}(x) d\mu_1 > \int \chi_{\max}(x) d\mu_2.$$

Let $h^* = \min\{h_{\mu_1}(f), h_{\mu_2}(f)\}$, $H^* = \max\{h_{\mu_1}(f), h_{\mu_2}(f)\}$.

Proposition 3.2.

$$\min\{h_{\text{top}}(MLI(A, f), f), h_{\text{top}}^P(MLI(A, f), f)\} \geq H^*, \quad h_{\text{top}}^B(MLI(A, f), f) \geq h^*.$$

Proof. From (2.5), $h_{\text{top}}^P(MLI(A, f), f) \leq h_{\text{top}}(MLI(A, f), f)$, thus by Proposition 2.12 we only need to show for any $\gamma > 0$,

$$h_{\text{top}}^P(MLI(A, f), f) \geq H^* - 5\gamma, \quad h_{\text{top}}^B(MLI(A, f), f) \geq h^* - 4\gamma.$$

Now fix $\gamma > 0$.

Step 1. Choice of separated sets.

If let $a = \int \chi_{\max}(x) d\mu_1$ and $b = \int \chi_{\max}(x) d\mu_2$, then $a - b > 0$. Let λ be the positive number in the definition of exponential shadowing. Take $\epsilon \in (0, \frac{a-b}{8})$ satisfying $\lambda > \epsilon/\alpha$.

Fix a number $\rho \in (0, 1)$. By Katok's definition of metric entropy, take $\varepsilon > 0$ such that

$$(3.1) \quad h_{\mu_i}(f, 4\varepsilon) = \liminf_{n \rightarrow +\infty} \frac{1}{n} \log N_n^{\mu_i}(4\varepsilon, \rho) > h_{\mu_i}(f) - \gamma, \quad i = 1, 2.$$

Take $\varsigma > 0$ small enough such that

$$(3.2) \quad h_{\mu_i}(f, 4\varepsilon) - (2 + 2\varsigma + h_{\mu_i}(f, 4\varepsilon))\varsigma \geq h_{\mu_i}(f, 4\varepsilon) - \gamma, \quad i = 1, 2.$$

For the measures μ_1 and μ_2 , take l large enough such that

$$\mu_1(\mathcal{R}_{\epsilon, l}^{\mu_1}) > 1 - \rho, \quad \mu_2(\mathcal{R}_{\epsilon, l}^{\mu_2}) > 1 - \rho.$$

Take $\eta > 0, \tau \in (0, \varepsilon)$ small enough such that it is applicable to Lemma 2.16 and Lemma 2.17. For τ, λ , by exponential shadowing there is $\delta > 0$ such that for any δ -pseudo-orbit $\{x_i, n_i\}_{i=-\infty}^{+\infty}$, there is a (τ, λ) -shadowing point $x \in X$ for $\{x_i, n_i\}_{i=-\infty}^{+\infty}$.

By compactness of X , we can choose a finite open cover $\{U_i\}_{i=1}^q$ with $\text{diam}(U_i) < \delta$. Define

$$T_{i,j} = \min\{n \mid U_i \cap f^{-n-p}U_j \neq \emptyset, \forall p \geq 0\}.$$

Since f is topologically mixing, $T_{i,j}$ is well-defined. Take $N = \max\{T_{i,j} \mid 1 \leq i, j \leq q\}$. Then for any $n \geq N$ and any $1 \leq i, j \leq q$,

$$(3.3) \quad U_i \cap f^{-n}U_j \neq \emptyset.$$

Let $\mu = \mu_i$ and $\Gamma = \mathcal{R}_{\epsilon, l}^{\mu_i}$ in Proposition 2.2. For above $\delta > 0, \varsigma > 0, \varepsilon > 0$, there is $T_* > 0$ such that for any large $N_* > 0$, there is $n_i \geq N_*$ and $(n_i, 4\varepsilon)$ -separated set $E_{n_i}^i \subseteq \mathcal{R}_{\epsilon, l}^{\mu_i}$ such that for each $i = 1, 2$,

- (1) for any $x \in E_{n_i}^i, f^{n_i}x \in \mathcal{R}_{\epsilon, l}^{\mu_i}$ and for any $x, y \in E_{n_i}^i, d(y, f^{n_i}x) < \delta$;
- (2) $\frac{1}{n_i} \log T_* \# E_{n_i}^i \geq h_\mu(f, \varepsilon) - (2 + 2\varsigma + h_\mu(f, \varepsilon))\varsigma$.

By (3.1) and (3.2), $\frac{1}{n_i} \log T_* \# E_{n_i}^i \geq h_{\mu_i}(f) - 2\gamma, i = 1, 2$. We emphasize that n_1 and n_2 can be chosen arbitrarily large. So we can take n_1, n_2 also satisfy following estimates: $n_1 \gg N$ such that

$$(3.4) \quad n_1 \gamma > (h^* - 3\gamma)N, \quad 2 < e^{n_1 \epsilon}, \quad \frac{1}{n_1} \log \# E_{n_1}^1 \geq h_{\mu_1}(f) - 3\gamma$$

and $n_2 \gg N$ large enough such that

$$(3.5) \quad n_2 \gamma > (h^* - 3\gamma)N, \quad l^2 e^l < e^{n_2 \epsilon}, \quad \frac{1}{n_2} \log \# E_{n_2}^2 \geq h_{\mu_2}(f) - 3\gamma.$$

Let $g : \mathbb{N} \rightarrow \{1, 2\}$ be given by $g(k) = (k + 1)(\text{mod } 2) + 1$ and let

$$\mathcal{S}_k := E_{n_{g(k)}}^{g(k)}, \quad n_k = n_{g(k)}, \quad \mu_k = \mu_{g(k)}.$$

From above construction, we have

$$(3.6) \quad \# \mathcal{S}_k \geq \exp(h_{\mu_k}(f) - 3\gamma)n_k;$$

and

$$(3.7) \quad \text{for any } x \in \mathcal{S}_k, f^{n_k}x \in \mathcal{R}_{\epsilon, l}^{\mu_k} \text{ and for any } x, y \in \mathcal{S}_k, d(y, f^{n_k}x) < \delta.$$

Step 2. Construction of the fractal F.

Let us choose a sequence with $N_0 = 0$ and N_k increasing to ∞ sufficiently quickly so that

$$(3.8) \quad \lim_{k \rightarrow \infty} \frac{n_{k+1}}{N_k} = 0, \quad \lim_{k \rightarrow \infty} \frac{N_1 n_1 + \dots + N_k n_k + kN}{N_{k+1}} = 0.$$

Let $\underline{x}_i = (x_1^i, \dots, x_{N_i}^i) \in S_i^{N_i}$. For any $(\underline{x}_1, \dots, \underline{x}_k) \in S_1^{N_1} \times \dots \times S_k^{N_k}$, by the exponential shadowing property, we have $B(\underline{x}_1, \dots, \underline{x}_k) :=$

$$\bigcap_{i=1}^k \bigcap_{j=1}^{N_i} f^{-\sum_{l=0}^{i-1} N_l n_l - (i-1)N - (j-1)n_i} \bigcap_{t=0}^{n_i-1} f^{-t} B(f^t x_j^i, \tau e^{\lambda \min\{t, n_i-t\}}) \neq \emptyset.$$

Let us explain this emptiness more precisely. Fix $(\underline{x}_1, \dots, \underline{x}_k) \in S_1^{N_1} \times \dots \times S_k^{N_k}$. Recall above finite open cover $\{U_t\}_{t=1}^q$ and we can take U_{t_i} and $U_{t_{i+1}}$ such that $f^{n_i} x_{N_i}^i \in U_{t_i}$ and $x_1^{i+1} \in U_{t_{i+1}}$. By (3.3), we can choose some $y_i \in U_{t_i} \cap f^{-N} U_{t_{i+1}}$. Then

$$d(f^{n_i} x_{N_i}^i, y) \leq \text{diam} U_{t_i} < \delta, \quad d(f^N y_i, x_1^{i+1}) \leq \text{diam} U_{t_{i+1}} < \delta.$$

By (3.7), we know that

$$d(f^{n_i} x_j^i, x_{j+1}^i) < \delta, \quad j = 1, 2, \dots, x_{N_i}, \quad i = 1, 2, \dots, k.$$

Then the following orbit segments form a δ -pseudo-orbit:

$$\begin{aligned}
& \dots, f^{-t-1}x_1^1, f^{-t}x_1^1, f^{-t+1}x_1^1, \dots, f^{-1}x_1^1, \\
& x_1^1, fx_1^1, \dots, f^{n_1-1}x_1^1, \quad x_2^1, \dots, f^{n_1-1}x_2^1, \quad \dots, \quad x_{N_1}^1, fx_{N_1}^1, \dots, f^{n_1-1}x_{N_1}^1, \\
& y_1, fy_1, f^2y_1, \dots, f^{N-1}y_1, \\
& x_1^2, fx_1^2, \dots, f^{n_2-1}x_1^2, \quad x_2^2, \dots, f^{n_2-1}x_2^2, \quad \dots, \quad x_{N_2}^2, fx_{N_2}^2, \dots, f^{n_2-1}x_{N_2}^2, \\
& y_2, fy_2, f^2y_2, \dots, f^{N-1}y_2, \\
& \dots, \\
& x_1^i, fx_1^i, \dots, f^{n_i-1}x_1^i, \quad x_2^i, \dots, f^{n_i-1}x_2^i, \quad \dots, \quad x_{N_i}^i, \dots, f^{n_i-1}x_{N_i}^i, \\
& y_i, fy_i, f^2y_i, \dots, f^{N-1}y_i, \\
& \dots, \\
& x_1^{k-1}, fx_1^{k-1}, \dots, f^{n_{k-1}-1}x_1^{k-1}, \quad \dots, \quad x_{N_{k-1}}^{k-1}, \dots, f^{n_{k-1}-1}x_{N_{k-1}}^{k-1}, \\
& y_{k-1}, fy_{k-1}, f^2y_{k-1}, \dots, f^{N-1}y_{k-1}, \\
& x_1^k, fx_1^k, \dots, f^{n_k-1}x_1^k, \quad x_2^k, \dots, f^{n_k-1}x_2^k, \quad \dots, \quad x_{N_k}^k, \dots, f^{n_k-1}x_{N_k}^k, \\
& f^{n_k}x_{N_k}^k, f^{n_k+1}x_{N_k}^k, \dots, f^tx_{N_k}^k, f^{t+1}x_{N_k}^k, \dots
\end{aligned}$$

By exponential shadowing, there is a point x whose orbit (τ, λ) -shadows above pseudo-orbit and thus $x \in B(\underline{x}_1, \dots, \underline{x}_k)$.

We define F_k by

$$F_k = \{\overline{B(\underline{x}_1, \dots, \underline{x}_k)} : (\underline{x}_1, \dots, \underline{x}_k) \in S_1^{N_1} \times \dots \times S_k^{N_k}\}.$$

Note that F_k is non-empty, compact and $F_{k+1} \subseteq F_k$. Define $F = \bigcap_{k=1}^{\infty} F_k$. Remark that F is compact and non-empty.

Lemma 3.1. For any $p \in F$, the sequence $\frac{1}{t_k} \log \|A(t_k, p)\|$ diverges, where $t_k = \sum_{i=0}^k N_i n_i + kN$.

Proof. Choose $p \in F$ and let $p_k := f^{t_{k-1}}p$. Then there exists $(x_1^k, \dots, x_{N_k}^k) \in S_k^{N_k}$ such that

$$p_k \in \bigcap_{j=1}^{N_k} f^{-(j-1)n_k} \overline{\bigcap_{t=0}^{n_k-1} f^{-t} B(f^t x_j^k, \tau e^{\lambda \min\{t, n_k-t\}})}.$$

We follow the idea of [25] to give following estimates. Firstly let us consider k with $k = 1 \bmod 2$. Fix $j = 1, 2, \dots, N_k$ and consider the orbit segments

$$p_{k+(j-1)n_k}, fp_{k+(j-1)n_k}, \dots, f^{n_k} p_{k+(j-1)n_k}$$

and $x_j^k, fx_j^k, \dots, f^{n_k} x_j^k$. Since the Lyapunov splitting and Lyapunov metric are continuous on the compact set $\mathcal{R}_{\epsilon, l}^{\mu_k}$, the cones $K_0^{\eta}(x_{j+1}^k) \subseteq K_0 = K_0(x_{j+1}^k)$ (centered at point x_{j+1}^k) and $K_0^{\eta}(f^{n_k} x_j^k)$ are close if x_{j+1}^k and $f^{n_k} x_j^k$ are close enough. Therefore we can ensure that $K_0^{\eta}(f^{n_k} x_j^k) \subset K_0(x_{j+1}^k)$ if δ small enough and thus $A(p_{k+(j-1)n_k}, n_k)(K_0(x_{j+1}^k)) \subset K_0(x_{j+1}^k)$. Using the norm estimate in Lemma 2.17 (in this estimate $\chi = a$, being the largest Lyapunov exponent of $\mu_k = \mu_1$) and (2.12), (2.19), we obtain for any $u \in K_0^{\eta}(x_j^k)$

$$\begin{aligned}
\|A(p_{k+(j-1)n_k}, n_k) u\|_{n_k} &\geq \|(A(p_{k+(j-1)n_k}, n_k) u)'\|_{n_k} \geq e^{n_k(a-2\epsilon)} \|u'\|_0 \\
&\geq \frac{1}{\sqrt{2}} e^{n_k(a-2\epsilon)} \|u\|_0 \geq \frac{1}{2} e^{n_k(a-2\epsilon)} \|u\|_{n_k}.
\end{aligned}$$

Since $A(p_{k+(j-1)n_k}, n_k)u \in K_0(x_{j+1}^k)$ for any $u \in K_0^\eta(x_j^k)$, we can iteratively apply $A(p, n)$ and use the inequality above to estimate the largest Lyapunov exponent at p

$$\|A(p_k, N_k n_k)u\|_{n_k} \geq \left(\left(\frac{1}{2} e^{n_k(a-2\epsilon)} \right)^{N_k} \|u\|_{n_k} \right).$$

Then we have

$$\frac{1}{n_k N_k} \|A(n_k N_k, p_k)\| \geq \frac{1}{n_k} \log \frac{1}{2} + a - 2\epsilon \geq a - 3\epsilon.$$

It follows that

$$\liminf_{k \rightarrow \infty} \frac{1}{n_k N_k} \|A(n_k N_k, p_k)\| \geq a - 3\epsilon.$$

We can use the fact that $\frac{n_k N_k}{t_k} \rightarrow 1$ to prove that for k with $k = 1 \pmod 2$,

$$(3.9) \quad \liminf_{k \rightarrow \infty} \frac{1}{t_k} \|A(t_k, p)\| \geq \liminf_{k \rightarrow \infty} \frac{1}{n_k N_k} \|A(n_k N_k, p_k)\| \geq a - 3\epsilon.$$

Secondly let us consider k with $k = 0 \pmod 2$. Fix $j = 1, 2, \dots, N_k$, for the orbit segments

$$p_{k+(j-1)n_k}, f p_{k+(j-1)n_k}, \dots, f^{n_k} p_{k+(j-1)n_k}$$

and $x_j^k, f x_j^k, \dots, f^{n_k} x_j^k$, by the first estimate in (2.18) of Lemma 2.16 (in this estimate $\chi = b$, being the largest Lyapunov exponent of $\mu_k = \mu_2$) we have

$$\|A(n_k, p_{k+(j-1)n_k})\| \leq l^2 e^l e^{n_k(b+\epsilon)}$$

Then we have

$$\|A(n_k N_k, p_k)\| \leq \prod_{j=1}^{N_k} \|A(n_k, p_{k+(j-1)n_k})\| \leq (l^2 e^l)^{N_k} \cdot e^{n_k N_k(b+\epsilon)}$$

and hence combining with $l^2 e^l < e^{n_2 \epsilon} = e^{n_k \epsilon}$ we have

$$\frac{1}{n_k N_k} \|A(n_k N_k, p_k)\| \leq \frac{1}{n_k} \log(l^2 e^l) + b + \epsilon \leq b + 2\epsilon.$$

It follows that

$$\limsup_{k \rightarrow \infty} \frac{1}{n_k N_k} \|A(n_k N_k, p_k)\| \leq b + 2\epsilon.$$

We can use the fact that $\frac{n_k N_k}{t_k} \rightarrow 1$ to prove that for k with $k = 0 \pmod 2$,

$$(3.10) \quad \limsup_{k \rightarrow \infty} \frac{1}{t_k} \|A(t_k, p)\| \leq \limsup_{k \rightarrow \infty} \frac{1}{n_k N_k} \|A(n_k N_k, p_k)\| \leq b + 2\epsilon.$$

Since the choice of ϵ satisfies $b + 2\epsilon < a - 3\epsilon$, by (3.9) and (3.10), the sequence $\frac{1}{t_k} \|A(t_k, p)\|$ diverges. \square

Step 3. Construction of a special sequence of measures ω_k . We must first undertake an intermediate construction. For each $\underline{x} = (x_1, \dots, x_k) \in S_1^{N_1} \times \dots \times S_k^{N_k}$, we choose one point $z = z(\underline{x})$ such that

$$z \in B(x_1, \dots, x_k).$$

Let \mathcal{T}_k be the set of all points constructed in this way. We show that points constructed in this way are distinct and thus $\#\mathcal{T}_k = \#S_1^{N_1} \dots \#S_k^{N_k}$.

Lemma 3.2. Let \underline{x} and \underline{y} be distinct members of $S_1^{N_1} \times \dots \times S_k^{N_k}$. Then $z_1 := z(\underline{x})$ and $z_2 := z(\underline{y})$ are distinct points. Thus $\#\mathcal{T}_k = \#S_1^{N_1} \dots \#S_k^{N_k}$.

Proof. Since $\underline{x} \neq \underline{y}$, there exists i, j so $x_j^i \neq y_j^i$. We have

$$d_{n_i}(x_j^i, f^h z_1) < \tau < \varepsilon \text{ and } d_{n_i}(y_j^i, f^h z_2) < \tau < \varepsilon,$$

where $h = \sum_{l=0}^{i-1} N_l n_l + i - 1N + (j - 1)n_i$. Since $d_{n_i}(x_j^i, y_j^i) > 4\varepsilon$, we have $d_{n_i}(f^h z_1, f^h z_2) > 2\varepsilon$. \square

Now we start to define the measures on F which yield the required estimates for the Entropy Distribution Principle. For each k , define a Dirac measure centred on \mathcal{T}_k . More precisely, let

$$\nu_k := \sum_{z \in \mathcal{T}_k} \delta_z$$

We normalize ν_k to get a sequence of probability measures ω_k , i.e. $\omega_k := \frac{1}{\#\mathcal{T}_k} \nu_k$.

Lemma 3.3. Suppose ω is a limit measure of the sequence of probability measures ω_k . Then $\omega(F) = 1$.

Proof. For any fixed l and $\forall p \geq 0$, $\omega_{l+p}(F_l) = 1$ since $\omega_{l+p}(F_{l+p}) = 1$ and $F_{l+p} \subseteq F_l$. Suppose $\omega = \lim_{k \rightarrow \infty} \omega_{l_k}$ for some $l_k \rightarrow \infty$, then $\omega(F_l) \geq \limsup_{k \rightarrow \infty} \omega_{l_k}(F_l) = 1$. It follows that $\omega(F) = \lim_{l \rightarrow \infty} \omega(F_l) = 1$. \square

Step 4. Estimate of $h_{top}^B(MLI(A, f), f)$.

Let $\mathcal{B} := B_n(q, \varepsilon)$ be an arbitrary ball which intersects F . Let k be the unique number which satisfies $t_k \leq n < t_{k+1}$.

We firstly consider n with $t_k \leq n < t_{k+1} - N$. Let $j \in \{0, \dots, N_{k+1} - 1\}$ be the unique number so

$$t_k + n_{k+1}j \leq n < t_k + n_{k+1}(j+1).$$

We assume that $j \geq 1$ and leave the details of the simpler case $j = 0$ to the reader. The following lemma reflects the restriction on the number of points that can be in $\mathcal{B} \cap \mathcal{T}_{k+p}$.

Lemma 3.4. For $p \geq 1$, $\omega_{k+p}(\mathcal{B}) \leq (\#\mathcal{T}_k)^{-1}(\#S_{k+1})^{-j}$

Proof. First we show that $\omega_{k+1}(\mathcal{B}) \leq (\#\mathcal{T}_k)^{-1}(\#S_{k+1})^{-j}$. We require an upper bound for the number of points in $\mathcal{T}_{k+1} \cap \mathcal{B}$. If $\omega_{k+1}(\mathcal{B}) > 0$, then $\mathcal{T}_{k+1} \cap \mathcal{B} \neq \emptyset$. Let $z = z(\underline{x}, \underline{x}_{k+1}) \in \mathcal{T}_{k+1} \cap \mathcal{B}$ where $\underline{x} \in S_1^{N_1} \times \dots \times S_k^{N_k}$ and $\underline{x}_{k+1} \in S_{k+1}^{N_{k+1}}$. Let

$$\mathcal{A}_{\underline{x}; x_1, \dots, x_j} = \{z(\underline{x}, y_1, \dots, y_{N_{k+1}}) \in \mathcal{T}_{k+1} : x_1 = y_1, \dots, x_j = y_j\}.$$

We suppose that $z' = z(\underline{y}, \underline{y}_{k+1}) \in \mathcal{T}_{k+1} \cap \mathcal{B}$ and show that $z' \in \mathcal{A}_{\underline{x}; x_1, \dots, x_j}$. We have $d_n(z, z') < 2\varepsilon$ and we show that this implies $x_l = y_l$ for $l \in \{1, 2, \dots, j\}$ (the proof that $\underline{x} = \underline{y}$ is similar). Suppose that $y_l \neq x_l$ and let $a_l = t_k + (l-1)(n_{k+1})$. Using

$$d_{n_{k+1}}(f^{a_l} z, x_l) < \tau < \varepsilon \text{ and } d_{n_{k+1}}(f^{a_l} z', y_l) < \tau < \varepsilon.$$

Recall that $d_{n_{k+1}}(x_l, y_l) > 4\varepsilon$. Then we have

$$\begin{aligned} d_n(z, z') &\geq d_{n_{k+1}}(f^{a_l} z, f^{a_l} z') \\ &\geq d_{n_{k+1}}(x_l, y_l) - d_{n_{k+1}}(f^{a_l} z, x_l) - d_{n_{k+1}}(f^{a_l} z', y_l) > 2\varepsilon, \end{aligned}$$

which is a contradiction. Thus, we have

$$\nu_{k+1}(\mathcal{B}) \leq \#\mathcal{A}_{\underline{x}; x_1, \dots, x_j} = (\#S_{k+1})^{N_{k+1}-j},$$

$$\omega_{k+1}(\mathcal{B}) \leq (\#\mathcal{T}_{k+1})^{-1}(\#S_{k+1})^{N_{k+1}-j} = (\#\mathcal{T}_k)^{-1}(\#S_{k+1})^{-j}$$

Now consider $\omega_{k+p}(\mathcal{B})$. Arguing similarly to above, we have

$$\nu_{k+p}(\mathcal{B}) \leq \#\mathcal{A}_{\underline{x}; x_1, \dots, x_j} (\#S_{k+2})^{N_{k+1}} \dots (\#S_{k+p})^{N_{k+p}}$$

The desired result follows from this inequality by dividing by $\#\mathcal{T}_{k+p}$. \square

By (3.6), the first estimate in (3.4) and the first estimate in (3.5), we have

$$\begin{aligned}
\#\mathcal{T}_k(\#\mathcal{S}_{k+1})^j &\geq \exp\{(h^* - 3\gamma)(N_1n_1 + N_2n_2 + \dots + N_kn_k + jn_{k+1})\} \\
&\geq \exp\{(h^* - 3\gamma)(n - kN)\} = \exp\{(h^* - 3\gamma)n \exp\{(3\gamma - h^*)kN\}\} \\
&= \exp\{(h^* - 4\gamma)n\} \exp\{n\gamma + (3\gamma - h^*)kN\} \\
&\geq \exp\{(h^* - 4\gamma)n\} \exp\left\{\sum_{i=1}^k n_i\gamma + (3\gamma - h^*)kN\right\} \\
&\geq \exp\{(h^* - 4\gamma)n\}.
\end{aligned}$$

$$(3.11) \quad \omega_{k+p}(\mathcal{B}) \leq (\#\mathcal{T}_{k-1})^{-1}(\#\mathcal{S}_k)^{-j} \leq \exp\{-(h^* - 4\gamma)n\}.$$

Now we consider n with $t_{k+1} - N \leq n < t_{k+1}$. By Lemma 3.4 and (3.11), using

$$B_{t_{k+1}-N-1}(q, \varepsilon) \supseteq \mathcal{B},$$

we have that for $p \geq 1$,

$$\omega_{k+p}(\mathcal{B}) \leq \omega_{k+p}(B_{t_{k+1}-N-1}(q, \varepsilon)) \leq \exp\{-(h^* - 4\gamma)(t_{k+1} - N - 1)\}.$$

Combining this with the first estimate in (3.4) and the first estimate in (3.5), we have

$$(3.12) \quad \omega_{k+p}(\mathcal{B}) \leq \exp\{-(h^* - 4\gamma)(t_{k+1} - N - 1)\} \leq \exp\{-(h^* - 5\gamma)n\}.$$

Combining (3.11) and (3.12) we have for all n ,

$$(3.13) \quad \limsup_{l \rightarrow \infty} \omega_l(B_n(q, \varepsilon)) \leq \exp\{-n(h^* - 5\gamma)\}.$$

By (3.13) and Entropy Distribution Principle (Proposition 2.1), $h_{top}^B(F, \varepsilon, f) \geq h^* - 5\gamma$. Since γ and ε were arbitrary and $F \subset MLI(A, f)$, we have

$$h_{top}^B(MLI(A, f), f) \geq h^*.$$

Remark that there is another way to get this estimate. More precisely, from (3.13) we know that $\omega(B_n(q, \varepsilon)) \leq \exp\{-n(h^* - 4\gamma)\}$. So for any $q \in F$,

$$\underline{h}_\mu(f, q) \geq h^* - 5\gamma.$$

By (2.4)

$$h_{top}^B(F, f) = \sup\{\underline{h}_\mu(f) : \mu \in \mathcal{M}(X), \mu(F) = 1\} \geq h^* - 5\gamma.$$

Step 5. Estimate of $h_{top}^P(MLI(A, f), f)$. By (3.6), the first estimate in (3.4) and the first estimate in (3.5), we have

$$\#\mathcal{T}_k \geq \#\mathcal{S}_k \geq \exp\{(h_{\mu_k}(f) - 3\gamma)N_kn_k\}.$$

Without loss of generality, assume that $H^* = h_{\mu_1}(f)$. Combining this with Lemma 3.4 ($n = t_k$) and $\lim_{k \rightarrow \infty} \frac{N_kn_k}{t_k} = 1$, for large enough k with $k = 1 \pmod 2$, we have

$$(3.14) \quad \limsup_{l \rightarrow \infty} \omega_l(B_{t_k}(q, \varepsilon)) \leq (\#\mathcal{T}_k)^{-1} \leq \exp\{-t_k(H^* - 4\gamma)\}.$$

This implies that for large enough k with $k = 1 \pmod 2$,

$$\omega(B_{t_k}(q, \varepsilon)) \leq \liminf_{l \rightarrow \infty} \omega_l(B_{t_k}(q, \varepsilon)) \leq \exp\{-t_k(H^* - 4\gamma)\}.$$

So for every $q \in F$,

$$\overline{h}_\mu(f, q) \geq H^* - 4\gamma.$$

Since F is compact, by (2.7)

$$h_{top}^P(F, f) = \sup\{\overline{h}_\mu(f) : \mu \in \mathcal{M}(X), \mu(F) = 1\} \geq H^* - 4\gamma.$$

Since γ and ε were arbitrary and $F \subset MLI(A, f)$, we have $h_{top}^P(MLI(A, f), f) \geq H^*$. Now we ends the proof of Proposition 3.2. \square

Remark 3.3. The construction of F in above proof is very cautious and mainly used to deal with h_{top}^B . If we only need to estimate h_{top}^P (and h_{top}) in Step 5, we can choose another (little) simple constructed way: one just modify N_k by $N_k \equiv 1$ and n_k by a sequence with $\lim_{k \rightarrow \infty} \frac{n_k}{n_{k+1}} = 0$. In other words, we only need to use every \mathcal{S}_k just once. One can follow above proof straightforward to estimate h_{top}^P (and h_{top}) and here we omit the details. This is based on we only need a uniform estimate for the measure of $B_n(q, \varepsilon)$ of a subsequence of $n = t_k$. However, we emphasize that the new way is not valid to deal with h_{top}^B , since we need a uniform estimate for the measure of $B_n(q, \varepsilon)$ of arbitrarily large n .

Now we continue the proof of Theorem 3.1. By (1) of Proposition 2.12 and Proposition 3.2,

$$\begin{aligned} h_{top}(f) &= \sup_{\mu, \nu \in \mathcal{M}_f^e(X)} \{\max\{h_\mu(f), h_\nu(f)\} \mid \chi_{max}(\mu, A) > \chi_{max}(\nu, A)\} \\ &\leq \min\{h_{top}(MLI(A, f), f), h_{top}^P(MLI(A, f), f)\}. \end{aligned}$$

It follows that

$$h_{top}(f) = h_{top}(MLI(A, f), f) = h_{top}^P(MLI(A, f), f).$$

It is not difficult to check that a topologically mixing system with shadowing satisfies specification property (c.f. [17, Proposition 23.20, Definition 21.1]). Since f is topologically mixing and has exponential shadowing (stronger than shadowing), then f has specification and thus by [34, Theorem 2.1] (or see [35], also see [18] for similar discussion), f has entropy-density property. By (2) of Proposition 2.12 and Proposition 3.2,

$$\begin{aligned} h_{top}^B(MLI(A, f), f) &\geq \sup_{\mu, \nu \in \mathcal{M}_f^e(X)} \{\min\{h_\mu(f), h_\nu(f)\} \mid \chi_{max}(\mu, A) > \chi_{max}(\nu, A)\} \\ &= \sup_{\mu \in \mathcal{M}_f^e(X)} \{h_\mu(f) \mid \chi_{max}(\mu, A) > \inf_{\nu \in \mathcal{M}_f(X)} \chi_{max}(\nu, A)\}. \end{aligned}$$

Now we complete the proof of Theorem 3.1. □

By Theorem 3.1 and (2) of Proposition 2.12, we have a direct consequence.

Corollary 3.4. *Let (X, d) be a compact metric space and $f : X \mapsto X$ be a topologically mixing homeomorphism with exponential shadowing property and let $A : M \rightarrow GL(m, \mathbb{R})$ be a Hölder continuous matrix function. If $\chi_{max}(A, \mu)$ is lower semi-continuous with respect to $\mu \in \mathcal{M}_f(X)$, then*

- (1) *either all ergodic measures have same Maximal Lyapunov exponents w.r.t. A , or*
- (2) *the entropy of maximal Lyapunov irregular set of A satisfies that*

$$h_{top}^B(MLI(A, f), f) = h_{top}(MLI(A, f), f) = h_{top}^P(MLI(A, f), f) = h_{top}(f).$$

3.2. Proof of Theorem 1.1 and 1.4. By Proposition 2.7, $f|_X$ has exponential shadowing. Then we can apply Theorem 3.1 to complete the proof of Theorem 1.1. Applying Theorem 1.1 for cocycle $A(x, n) = D_x f^n$, one ends the proof of Theorem 1.4. □

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REFERENCES

- [1] S. Albeverio, M. Pratsiovytyi and G. Torbin, *Topological and fractal properties of subsets of real numbers which are not normal*, Bull. Sci. Math., 129 (2005), 615-630.
- [2] I.-S. Baek and L. Olsen, *Baire category and extremely non-normal points of invariant sets of IFS's*, Discrete Contin. Dyn. Syst., 27 (2010), 935-943.
- [3] L. Barreira, *Dimension and recurrence in hyperbolic dynamics*, Progress in Mathematics, vol. 272, Birkhäuser, 2008.
- [4] L. Barreira, K. Gelfert, *Dimension estimates in smooth dynamics: a survey of recent results*, Ergodic Theory and Dynamical Systems, 2011, 31(03): 641-671.
- [5] Luis Barreira and Yakov B. Pesin, *Nonuniform hyperbolicity*, Cambridge Univ. Press, Cambridge (2007).
- [6] J. Bochi, M. Rams, *The entropy of Lyapunov-optimizing measures of some matrix cocycles*, arXiv preprint arXiv:1312.6718, 2013.
- [7] L. Barreira and J. Schmeling, *Sets of "non-typical" points have full topological entropy and full Hausdorff dimension*, Israel J. Math. 116 (2000), 29-70.
- [8] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Springer, Lecture Notes in Math. 470 (1975).
- [9] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc. 153, 1971, 401-414.
- [10] R. Bowen, *Topological entropy for noncompact sets*, Trans. Amer. Math. Soc. 184 (1973), 125-136.
- [11] R. Bowen, *Periodic orbits for hyperbolic flows*, Amer. J. Math., 94 (1972), 1-30.
- [12] R. Bowen, *Periodic points and measures for Axiom A diffeomorphisms*, Trans. Amer. Math. Soc. 154 (1971), 377-397.
- [13] T. Bousch, O. Jenkinson, *Cohomology classes of dynamically non-negative C^k functions*, Invent. Math. 148 (2002) 207-217.
- [14] M. Brin and A. Katok, *On local entropy. Geometric dynamics* (Rio de Janeiro, 1981), 30-38, Lecture Notes in Math., 1007, Springer, Berlin, 1983.
- [15] G. Contreras, A. O. Lopes and P. Thieullen, *Lyapunov minimizing measures for expanding maps of the circle*, Ergod. Th. & Dynam. Sys. 21 (2001), 1379-1409.
- [16] X. Dai, *Exponential closing property and approximation of Lyapunov exponents of linear cocycles*, Forum Mathematicum, 2011, 23(2): 321-347.
- [17] M. Denker, C. Grillenberger and K. Sigmund, *Ergodic Theory on the Compact Space*, Lecture Notes in Mathematics 527.
- [18] A. Eizenberg, Y. Kifer and B. Weiss, *Large Deviations for \mathbb{Z}^d -actions*, Commun. Math. Phys., 164, 433-454 (1994).
- [19] D. Feng, W. Huang, *Variational principles for topological entropies of subsets*, Journal of Functional Analysis, 2012, 263(8): 2228-2254.
- [20] A. Furman, *On the multiplicative ergodic theorem for uniquely ergodic systems*, Annales de l'Institut Henri Poincaré (B) Probability and Statistics, 1997, 33(6): 797-815.
- [21] J. Hyde, V. Laschos, L. Olsen, I. Petrykiewicz and A. Shaw, *Iterated Cesaro averages, frequencies of digits and Baire category*, Acta Arith., 144 (2010), 287-293.
- [22] A. Gogolev, *Diffeomorphisms Hölder conjugate to Anosov diffeomorphisms*, Ergodic Theory and Dynamical Systems, Vol. 30, no. 2 (2010), 441-456
- [23] B. Hasselblatt, Z. Nitecki and J. Propp, *Topological entropy for nonuniformly continuous maps*, Discrete Contin. Dyn. Syst. 22 (2008), 201-213.
- [24] M. Herman, *Construction d'un difféomorphisme minimal d'entropie topologique non nulle*, Ergod. Th. Dynam. Sys. 1 (1981), 65-76.
- [25] B. Kalinin, *Yuzvinsky Theorem for matrix cocycles*, Annals of mathematics, 2011, 173 (2), 1025-1042.
- [26] A. Katok, *Lyapunov exponents, entropy and periodic orbits for diffeomorphisms*, Inst. Hautes Etudes Sci. Publ. Math. 51 (1980), 137-173.
- [27] D. Lenz, *Existence of non-uniform cocycles on uniquely ergodic systems*, Annales de l'Institut Henri Poincaré (B) Probability and Statistics, 2004, 40(2): 197-206.
- [28] L. Mendoza, *Ergodic attractors for diffeomorphisms of surfaces*, J. London Math. Soc. **37** (1988), no. 2, 362-374.
- [29] I. D. Morris, *Ergodic optimization for generic continuous functions*, Discrete & Cont. Dyn. Sys, 2010, 27: 383-388.
- [30] L. Olsen, *Extremely non-normal numbers*, Math. Proc. Cambridge Philos. Soc., 137 (2004), 43-53.
- [31] K. Oliveira and X. Tian, *Non-uniform hyperbolicity and non-uniform specification*, Transactions of the American Mathematical Society, 365 (8), 2013, 4371-4392.

- [32] P. Oprocha and G. Zhang, *Dimensional entropy over sets and fibres*, Nonlinearity 24 (2011), no. 8, 2325–2346.
- [33] V. I. Oseledec, *Multiplicative ergodic theorem, Liapunov characteristic numbers for dynamical systems*, Trans. Moscow Math. Soc., 19 (1968), 197-221; translated from Russian.
- [34] C. Pfister, W. Sullivan, *Large Deviations Estimates for Dynamical Systems without the Specification Property. Application to the β -shifts*, Nonlinearity 18, 237-261 (2005).
- [35] C. Pfister, W. Sullivan, *On the topological entropy of saturated sets*, Ergod. Th. Dynam. Sys. 27, 929-956 (2007).
- [36] Y. B. Pesin, *Dimension Theory in Dimensional Systems: Contemporary Views and Applications*, University of Chicago Press, Chicago, 1997.
- [37] M. Pollicott, *Lectures on ergodic theory and Pesin theory on compact manifolds*, Cambridge Univ. Press, Cambridge (1993).
- [38] B. Saussol, S. Troubetzkoy and S. Vaienti, *Recurrence and Lyapunov exponents*, Mosc. Math. J. , Volume 3, Number 1 (2003) 189-203.
- [39] F. Takens, E. Verbitskiy, *On the variational principle for the topological entropy of certain non-compact sets*, Ergodic theory and dynamical systems, 2003, 23 (1): 317-348.
- [40] D. Thompson, *The irregular set for maps with the specification property has full topological pressure*, Dyn. Syst. 25 (2010), no. 1, 25-51.
- [41] D. Thompson, *Irregular sets, the β -transformation and the almost specification property*, Transactions of the American Mathematical Society, 2012, 364 (10): 5395-5414.
- [42] P. Walters, *Unique ergodicity and random matrix products*, Lyapunov Exponents, Springer Berlin Heidelberg, 1986: 37-55.
- [43] P. Walters, *An introduction to ergodic theory*, Springer-Verlag, 2001.
- [44] Z. Wang and W. Sun, *Lyapunov exponents of hyperbolic measures and hyperbolic period orbits*, Trans. Amer. math. Soc., 362 (2010), 4267-4282.
- [45] G. Yuan and B. R. Hunt. *Optimal orbits of hyperbolic systems*, Nonlinearity 12 (1999), 1207-1224.

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